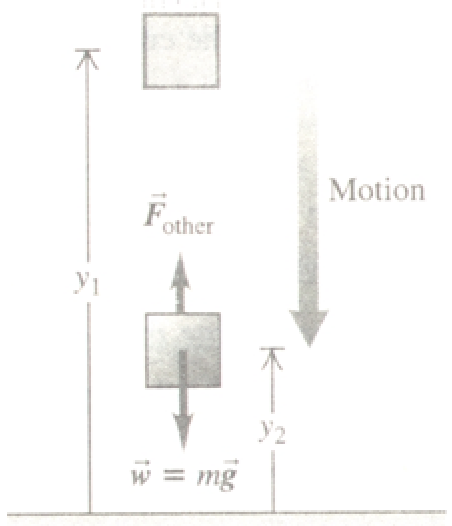
8.1
Gravitational Potential Energy

A particle gains or loses kinetic energy because it interacts with other objects that exert forces on it. During any interaction, the change in particle's kinetic energy is equal to the total work done on the particle by the forces that act on it.

In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work on the hammerhead to lift it. In hoisting the hammerhead into the air you are storing energy in the system, energy that is later converted into kinetic energy as the hammerhead falls.

This example points to the idea of an energy associated with the position of bodies in a system. This kind of energy is a measure of the potential or possibility for work to be done; when a hammerhead is at a raised position in the air, there is a potential for work to be done on it by the gravitational force, but only if the hammerhead is allowed to fall to the ground. For this reason, energy associated with position is called **potential energy**. The potential energy, associated with a body's weight and its height above the ground, is called *gravitational potential energy*.

Consider a body with mass *m* that moves along the *y*-axis as shown below.



The forces acting on it are its weight, with magnitude $w = mg$, and possibly some other forces; we call the vector sum (resultant) of all other forces $\mathbf{F}_{\text{other}}$. We want to find the work done by the weight when the body drops from a height y_1 above the origin to a lower height y_2. The weight and displacement are in the same direction, so the work W_{grav} done on the body by its weight is positive;

$$W_{\text{grav}} = Fs = mg(y_1 - y_2) = mgy_1 - mgy_2 \qquad (1)$$

Equation (1) shows that we can express W_{grav} in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity, the product of the weight mg and the height *y* above the origin of coordinates, is called the **gravitational potential energy**, *U*:

$$U = mgy \qquad (2)$$

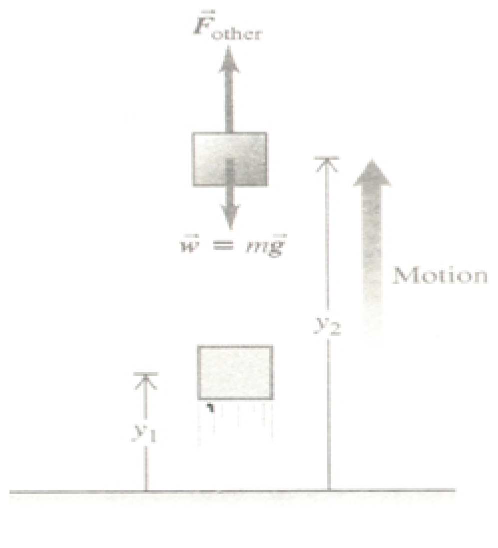
Its initial value is $U_1 = mgy_1$ and its final value is $U_2 = mgy_2$. The change in *U* is the final value minus the initial value, $\Delta U = U_2 - U_1$. We can express the work W_{grav} done by the gravitational force during the displacement from y_1 to y_2 as

$$W_{\text{grav}} = U_1 - U_2 = -(U_2 - U_1) = -\Delta U \qquad (3)$$

The negative sign in front of ΔU is *essential*.

When the body moves down, *y* decreases, the gravitational force does positive work and the gravitational potential energy decreases (ΔU < 0).

When the body moves up, *y* increases, the work done by the gravitational force is negative, and the gravitational potential energy increases (ΔU > 0).



Equation (3) shows that the unit of potential energy is the joule (J), the same unit as is used for work.

Gravitational potential energy is a shared property of the body and the earth. It increases if the earth stays fixed and the height of the body

$$K_1 + U_1 = K_2 + U_2 \qquad (4)$$

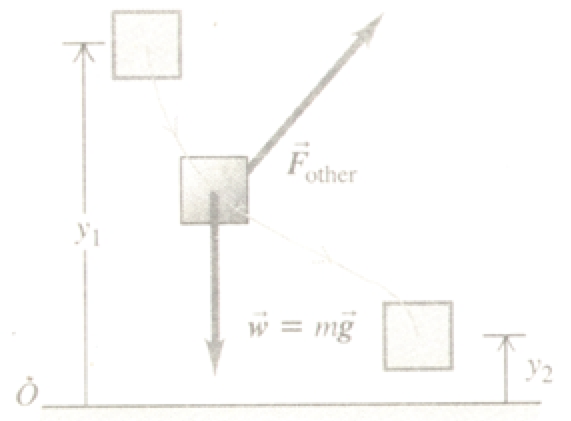
$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2$$

8.2

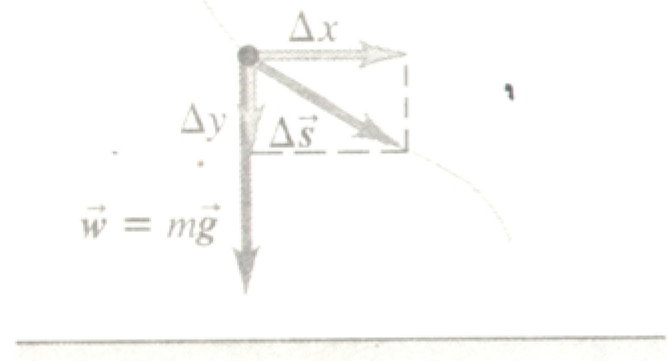
Motion along a Curved Path—Gravitational Potential Energy

When other forces act on the body in addition to its weight, then **F**$_{other}$ is not zero. For the pile driver, described in an earlier module, the force applied by the hoisting cable and the friction with the vertical guiding rails are examples of forces that might be included in **F**$_{other}$. We call the work done by **F**$_{other}$ as W_{other}.

Suppose the body moves along a curve as shown below. The body is acted on by the gravitational force $\mathbf{w} = m\mathbf{g}$ and by other forces whose resultant is $\mathbf{F}_{other}$.



To find the work done by the gravitational force during this displacement, we divide the path into small segments $\Delta\mathbf{s}$; a typical segment is shown below.



The work done by the gravitational force over this segment is the scalar product of the force and the displacement. In terms of unit vectors, the force is $\mathbf{w} = m\mathbf{g} = -mg\mathbf{j}$ and the displacement is $\Delta\mathbf{s} = \Delta x\mathbf{i} + \Delta y\mathbf{j}$, so the work done by the gravitational force is

$$\mathbf{w}\cdot\Delta\mathbf{s} = -mg\mathbf{j}\cdot(\Delta x\mathbf{i} + \Delta y\mathbf{j}) = -mg\Delta y$$

The work done by gravity is the same as though the body has been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the total work done by the gravitational force is $-mg$ multiplied by the total vertical displacement $(y_2 - y_1)$:

$$W_{grav} = -mg(y_2 - y_1) = mgy_1 - mgy_2 = U_1 - U_2$$

This is the same equation which we obtained earlier when the body moved along straight vertical line. So even if the path between two points is curved, the total work done by the gravitational force depends only on the difference in height between the two points of the path. The work is unaffected by any horizontal motion that may occur.

If the work W_{other} done by other forces $\mathbf{F}_{other}$ is zero, it is only the force of gravity that does work. As we have seen in the preceding module, the total mechanical energy is conserved, when only the force of gravity does work.

$$K_1 + U_1 = K_2 + U_2$$

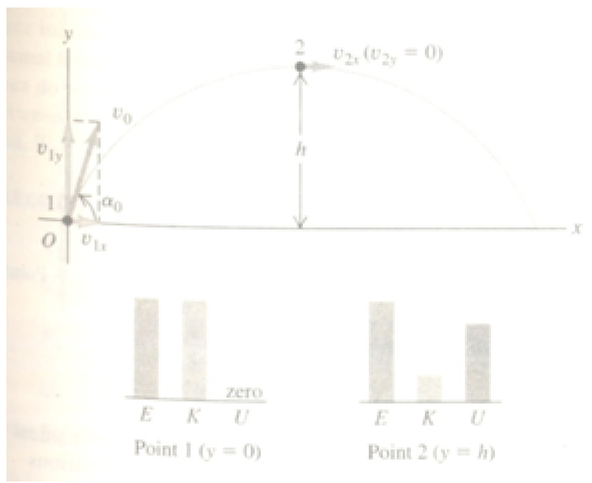
$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2$$

Example

Derive an expression for the maximum height *h* of a projectile launched with initial speed v_0 at an initial angle α_0.

Solution

Let point 1 at *y* = 0 be the launch point, where the speed is $v_1 = v_0$, and let point 2 at *y* = *h* be the highest point on the trajectory.



We can express the kinetic energy at each point in terms of the components of velocity, using $v^2 = v_x^2 + v_y^2$.

$$K_1 = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(v_{1x}^2 + v_{1y}^2\right).$$

$$K_2 = \frac{1}{2}m\left(v_{2x}^2 + v_{2y}^2\right)$$

Since the *x*-component of the acceleration is zero $v_{2x} = v_{1x}$. At the highest point of trajectory, the *y*-component of velocity is zero; so $v_{2y} = 0$.

Conservation of energy gives

$$K_1 + U_1 = K_2 + U_2$$

$$\frac{1}{2}m\left(v_{1x}^2 + v_{1y}^2\right) + 0 = \frac{1}{2}m\left(v_{2x}^2 + v_{2y}^2\right) + mgh$$

Using the relations $v_{2x} = v_{1x}$ and $v_{2y} = 0$, we have

$$v_{1y}^2 = 2gh$$

$$h = \frac{v_{1y}^2}{2g} = \frac{v_0^2\sin^2\alpha_0}{2g}$$

Thus,

$$h = \frac{v_0^2\sin^2\alpha_0}{2g}$$

8.3

Mechanical Energy

The body is acted on by the gravitational force $w = mg$ and possibly by other forces whose resultant is $\mathbf{F}_{other}$. The gravitational work is still given by $W_{grav} = U_1 - U_2 = -(U_2 - U_1) = -\Delta U$, but the total work W_{tot} is then sum of W_{grav} and the work done by $\mathbf{F}_{other}$. We call this additional work W_{other}, so the total work done by all forces is $W_{tot} = W_{grav} + W_{other}$. Equating this to the change in kinetic energy, we have

$$W_{other} + W_{grav} = K_2 - K_1$$
$$W_{other} + U_1 - U_2 = K_2 - K_1$$

which we can arrange in the form

$$K_1 + U_1 + W_{other} = K_2 + U_2$$

Using the appropriate expressions for the various energy terms, we obtain

$$\frac{1}{2}mv_1^2 + mgy_1 + W_{other} = \frac{1}{2}mv_2^2 + mgy_2$$

The work done by all forces other than the gravitational force equals the change in the total mechanical energy $E = K + U$ of the system, where U is the gravitational potential energy.

$$W_{other} = (K_2 - K_1) + (U_2 - U_1)$$

This is known as the *work-energy equation*.

The work done by all forces other than the gravitational force equals the change in the total mechaical energy $E = K + U$ of the system, where U is the gravitational potential energy and K is the kinetic energy. When W_{other} is positive, E increases, and $K_2 + U_2$ is greater than $K_1 + U_1$. When W_{other} is negative, E decreases. In the special case in which no forces other than the body's weight do work, $W_{other} = 0$. The total mechanical energy is then constant.

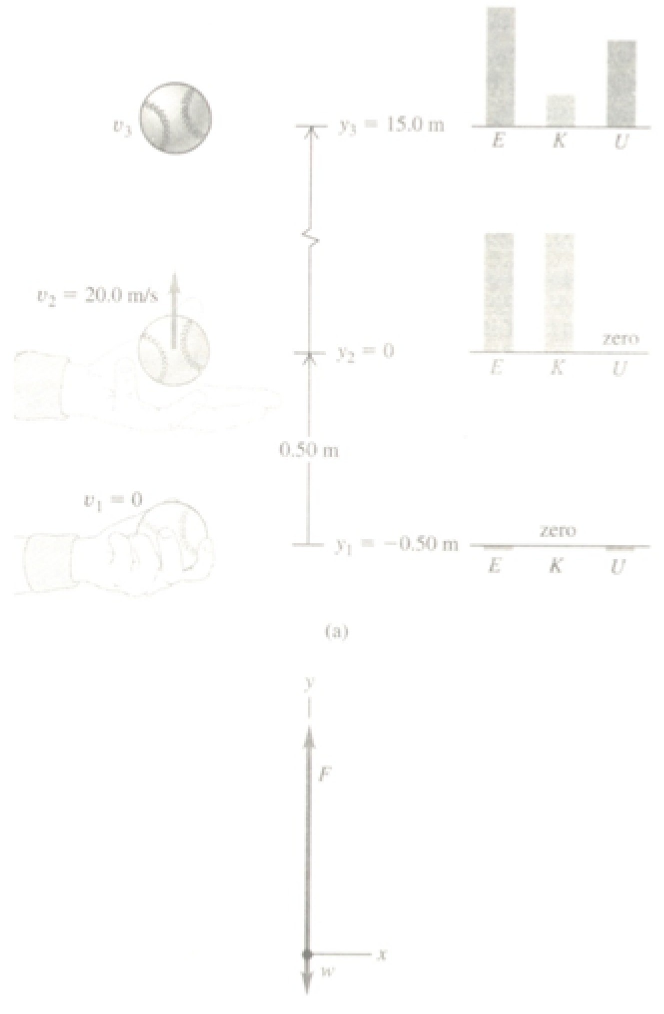
Example

You throw a 0.145-kg ball straight up in the air. Suppose your hand moves up 0.50 m while you are throwing the ball, which leaves your hand with an upward velocity of 20.0 m/s. Ignore air resistance.

1. Assuming that your hand exerts a constant upward force on the ball, find the magnitude of that force.
2. Find the speed of the ball at a point 15.0 m above the point where it leaves your hand.

Solution

In this example, we must consider the nongravitational work done by your hand. The figure below shows a free-body diagram for the ball while it is being thrown.



The ball's motion occurs in two stages: while it is in contact with your hand and after it leaves your hand. We let point 1 be where your hand first starts to move, point 2 be where the ball leaves your hand, and point 3 be where the ball is 15.0 m above point 2. The nongravitational force **F** of your hand acts only between points 1 and 2. We have $y_1 = -0.50\,\text{m}$, $y_2 = 0$, and $y_3 = 15.0\,\text{m}$. The ball starts at rest at point 1, so $v_1 = 0$, and we are given that the ball's speed as it leaves your hand is $v_2 = 20.0\,\text{m/s}$.

We have

$$K_1 = 0 \qquad U_1 = mgy_1 = 0.145 \times 9.80 \times (-0.50) = -0.71\,\text{J}$$

$$K_2 = \frac{1}{2}mv_2^2 = \frac{1}{2} \times 0.145 \times 20.0^2 = 29.0\,\text{J}$$

$$U_2 = mgy_2 = 0.145 \times 9.80 \times (0) = 0$$

The initial potential energy U_1 is negative because the ball was initially below the origin.

$$W_{other} = (K_2 - K_1) + (U_2 - U_1) = (29.0 - 0) + (0 - (-0.71)) = 29.71\,\text{J}$$

The kinetic energy of the ball increases by 29.0 J, and the potential energy increases by 0.71 J; the sum is $E_2 - E_1$, the change in total mechanical energy, which is equal to W_{other}.

$$W_{other} = F(y_2 - y_1) \qquad F = \frac{W_{other}}{y_2 - y_1} = \frac{29.71}{0.50} = 59.41\,\text{N}$$

We note that between points 2 and 3, total mechanical energy is conserved; the force of your hand no longer acts, so $W_{other} = 0$.

$$K_2 + U_2 = K_3 + U_3$$

$$U_3 = mgy_3 = 0.145 \times 9.80 \times (15.0) = 21.3\,\text{J}$$

$$K_3 = (K_2 + U_2) - U_3 = (29.0 + 0) - 21.3 = 7.7\,\text{J}$$

$$K_3 = \frac{1}{2}mv_{3y}^2 \qquad v_{3y} = \pm\sqrt{\frac{2K_3}{m}} = \pm\sqrt{\frac{2(7.7)}{0.145}} = \pm 10\,\text{m/s}$$

The significance of the plus-or-minus sign is that the ball passes point 3 twice, once on the way up and again on the way down.

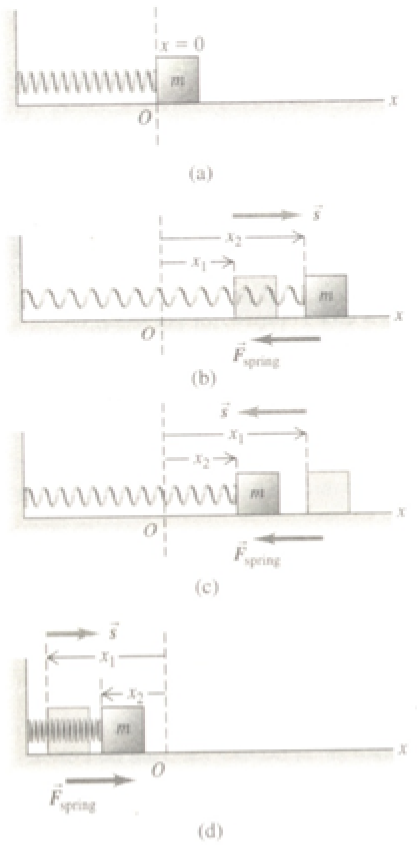
8.4

Elastic Potential Energy

When a railroad car runs into a spring bumper at the end of the track, the spring is compressed as the car is brought to a stop. If there is no friction, the bumper springs back and the car moves away in the opposite direction with its original speed. During the interaction with the spring, the car's kinetic energy has been "stored" in the elastic deformation of the string. We do work on the system to store energy, which can later be converted to kinetic energy.

A body is called *elastic* if it returns to its original shape and size after being deformed. We will consider an ideal spring. To keep such an ideal spring stretched by a distance x, we must exert a force $F = kx$, where k is the force constant of the spring.

Figure below shows a spring, with its left end stationary, and its right end attached to a block with mass *m* that can move along the *x*-axis.



In figure (a) the body is at x = 0 when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, and then let it go. As the block moves from one position x_1 to another position x_2, we found earlier in the preceding unit that the work we must do on the spring is

$$W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

where k is the force constant of the spring. We need the work done by the spring. From Newton's third law the two quantities of work are just negatives of each other. Therefore, we find that in a displacement from x_1 to x_2 the spring does an amount of work W_{el} given by

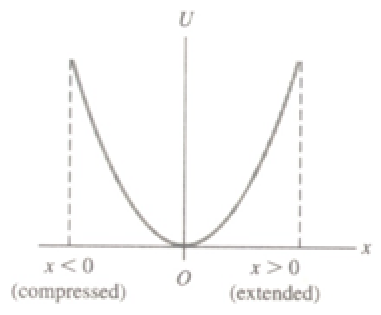
$$W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$

When x_1 and x_2 are both positive and $x_2 > x_1$ (figure b), the spring does negative work on the block, which moves in the $+x$-direction while the spring pulls on it in the $-x$-direction. The spring stretches farther, and the block slows down. When x_1 and x_2 are both positive and $x_2 < x_1$ (figure c), the spring does positive work as it relaxes and the block speeds up. Figure (d) is a compressed spring, and it does positive work on the block as it relaxes.

We define the ***elastic potential energy*** as

$$U = \frac{1}{2}kx^2$$

Figure below is a graph of this equation.



We can now express the work W_{el} done on the block by the elastic force in terms of the change in potential energy:

$$W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 = U_1 - U_2 = -\Delta U$$

When a stretched spring is stretched further, as in figure (b), W_{el} is negative and U increases; a greater amount of elastic potential energy is stored in the spring. When a stretched spring relaxes, as in figure (c), W_{el} is positive, and U decreases; the spring loses elastic potential energy. Negative values of *x* refer to a compressed spring. But, U is positive for both positive and negative *x*.

Gravitational potential energy is a shared property of a body and the earth, but elastic potential energy is stored just in the spring (or other deformable body). An important difference between gravitational potential energy $U = mgy$ and elastic potential energy $U = \frac{1}{2}kx^2$ is that we do not have the freedom to choose $x = 0$ to be wherever we wish. To be consistent with $U = \frac{1}{2}kx^2$, $x = 0$ must be the position at which the spring is neither stretched nor compressed. At that position, its elastic potential energy and force that it exerts are both zero.

If the elastic force is the only force that does work on the body, then

$$W_{tot} = W_{el} = U_1 - U_2$$

The work-energy theorem says that $W_{tot} = K_2 - K_1$, no matter what kind of forces are acting on a body. Therefore, it gives us

$$K_1 + U_1 = K_2 + U_2$$

So,

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$

In this case the total mechanical energy $E = K + U$ (the sum of kinetic and elastic potential energy) is conserved.

8.5

Work and Energy

Again, we will consider the spring as in the precding module. If forces other than the elastic force also do work on the body, we call their work W_{other}, as before. Then the total work is $W_{\text{tot}} = W_{\text{el}} + W_{\text{other}}$, and the work-energy theorem gives

$$W_{\text{el}} + W_{\text{other}} = K_2 - K_1$$

$$U_1 - U_2 + W_{\text{other}} = K_2 - K_1$$

$$K_1 + U_1 + W_{\text{other}} = K_2 + U_2$$

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 + W_{\text{other}} = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$

This equation shows that the work done by all forces other than the elastic force equals the change in the total mechanical energy $E = K + U$ of the system, where *U* is the elastic potential energy. "The system" is made up of the body of mass *m* and the spring of force constant *k*. When W_{other} is positive, E increases; when W_{other} is negative, E decreses.

When we have both gravitational and elastic forces, such as a block attached to the lower end of a vertically hanging spring, then

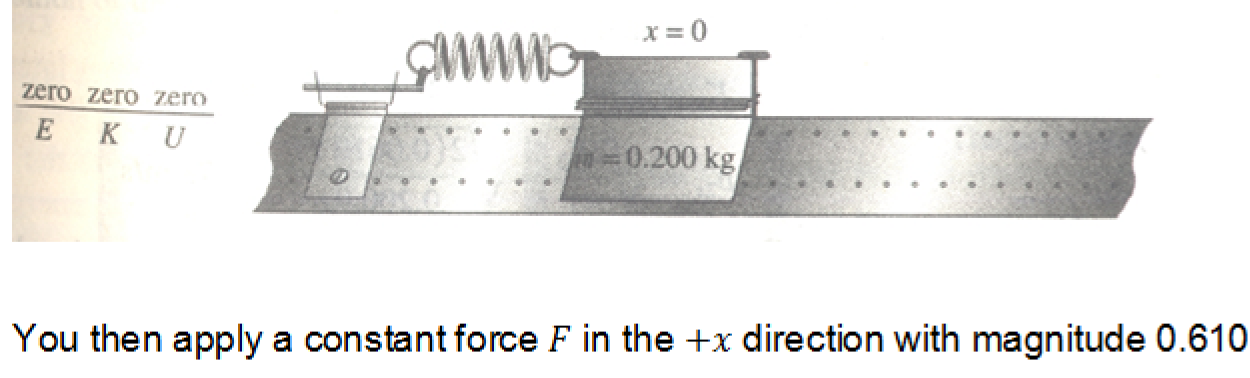
$$K_1 + U_{\text{grav},1} + U_{\text{el},1} + W_{\text{other}} = K_2 + U_{\text{grav},2} + U_{\text{el},2}$$

The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system, where *U* is the sum of the gravitational potential energy and the elastic potential energy. If the gravitational and elastic forces are the only forces that do work on the body, then W_{other} = 0 and the total mechanical energy (including both gravitational and elastic potential energy) is conserved.

In problem solving, we remember that the work done by the gravitational and elastic forces is accounted for by their potential energies; the work of the other forces, W_{other}, has to be included separately. Furthermore, in the elastic potential energy, $U_{\text{el}} = \frac{1}{2}kx^2$, *x* is the displacement of the spring from its unstretched length.

Example 1

In the figure below, a glider with mass $m = 0.200\text{ kg}$ sits on a frictionless horizontal air track, connected to a spring with force constant $k = 5.00\text{ N/m}$. The glider is initially at rest at $x = 0$, with the spring unstretched.



You then apply a constant force F in the $+x$ direction with magnitude 0.610 N to the glider. What is the glider's velocity when it has moved to $x = 0.100$ m?

Solution

Although the force **F** you apply is constant, the spring force isn't, so the acceleration of the glider won't be constant. Total mechanical energy is not conserved because of the work done by the force **F**. We use the energy relation.

Let point 1 be at $x = 0$, where the velocity is $v_{1x} = 0$, and let point 2 be at $x = 0.100\text{ m}$.

The energy quantities are

$$\begin{aligned} K_1 &= 0 \\ U_1 &= 0 \\ K_2 &= \frac{1}{2}mv_{2x}^2 \\ U_2 &= \frac{1}{2}(5.00)(0.100)^2 = 0.0250\text{ J} \\ W_{\text{other}} &= (0.610)(0.100) = 0.0610\text{ J} \end{aligned}$$

We use the energy relation.

$$\begin{aligned} K_1 + U_1 + W_{\text{other}} &= K_2 + U_2 \\ K_2 &= K_1 + U_1 - U_2 + W_{\text{other}} \\ &= 0 + 0 - 0.0250 + 0.0610 = 0.0360\text{ J} \\ v_{2x} &= \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2\times 0.0360}{0.200}} = 0.60\text{ m/s} \end{aligned}$$

We use the positive square root because the glider is moving in the +*x*-direction.

Example 2

In the example 1, suppose the force **F** is removed when the glider reaches the point $x = 0.100\text{ m}$. How much farther does the glider move before coming to rest?

Solution

After F is removed, the spring force is the only force doing work. Hence for this part of motion the mechanical energy $E = K + U$ is conserved.

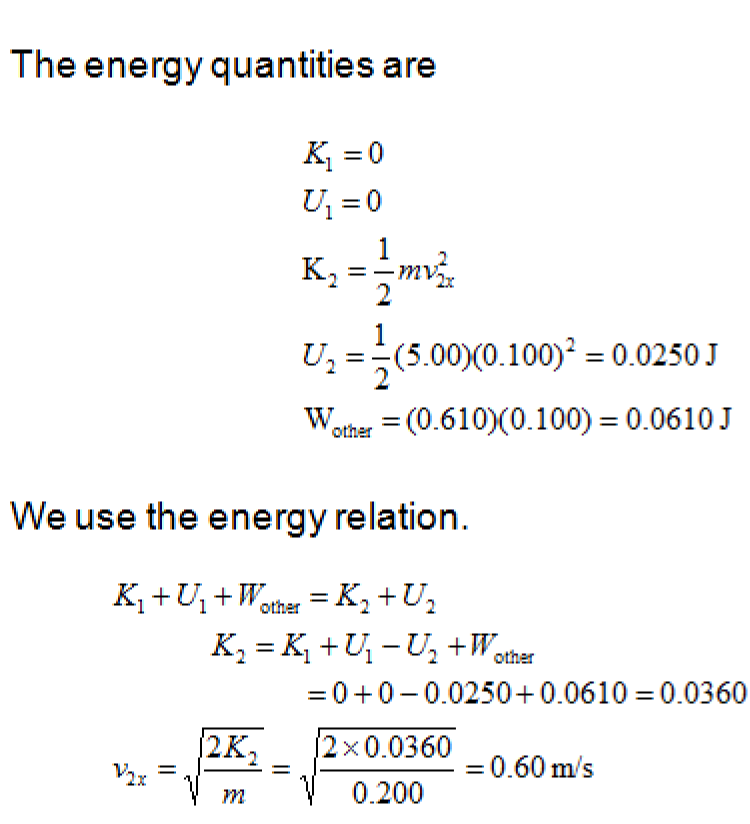
We'll let the point 2 be at $x = 0.100\text{ m}$, and let point 3 be where the glider comes instantaneously to rest. We'll use the conservation of energy expressions.

$$\begin{aligned} K_2 + U_2 &= K_3 + U_3 \\ U_3 &= K_2 + U_2 - K_3 \\ &= 0.0360 + 0.0250 - 0 = 0.0610\text{ J} \\ x_3 &= \sqrt{\frac{2U_3}{k}} = \sqrt{\frac{2\times 0.0610}{5.00}} = 0.156\text{ m} \end{aligned}$$

The body moves an additional 0.056 m after the force is removed at $x = 0.100\text{ m}$.

Example 3

In a "worst-case" design scenario, a 2000-kg elevator with broken cables is falling at 25 m/s when it first contacts a cushioning spring at the bottom of the shaft. The spring is supposed to stop the elevator, compressing 3.00 m as it does so.



During the motion a safety clamp applies a constant 17,000-N frictional force to the elevator. As a design consultant, you are asked to determine what the force constant of the spring should be?

Solution

We'll use the energy approach to determine the force constant. Total mechanical energy is not conserved because the friction force does negative work W_{other} on the elevator. This problem involves both gravitational and elastic potential energy.

We take point 1 as the position of the bottom of the elevator when it initially contacts the spring, and take point 2 as its position when it is at rest. We choose the origin to be at point 1, so $y_1 = 0$ and $y_2 = -3.00\text{ m}$. With this choice the coordinate of the upper end of the spring is the same as the coordinate of the elevator, so the elastic potential energy at any point between point 1 and point 2 is $U_{\text{el}} = \frac{1}{2}ky^2$. The gravitational potential energy is $U_{\text{grav}} = mgy$. We know the initial and final speeds of the elevator and the magnitude of the friction force.

$$\begin{aligned} K_1 &= \frac{1}{2}mv_1^2 = \frac{1}{2}(2000)(25)^2 = 625{,}000 \\ K_2 &= 0 \\ U_1 &= mgy_1 + \frac{1}{2}ky_1^2 = 0 + 0 = 0 \\ U_2 &= mgy_2 + \frac{1}{2}ky_2^2 \end{aligned}$$

The gravitational potential energy at point 2 is

$$mgy_2 = (2000)(9.80)(-3.00) = -58{,}800\text{ J}$$

The 17,000-N friction force, acting opposite to the 3.00-m displacement, does the work

$$W_{\text{other}} = -(17{,}000)(3.00) = -51{,}000\text{ J}$$

$$\begin{aligned} K_1 + U_1 + W_{\text{other}} &= K_2 + U_2 \\ K_1 + 0 + W_{\text{other}} &= 0 + \left(mgy_2 + \frac{1}{2}ky_2^2\right) \end{aligned}$$

So, the force constant of the spring is

$$\begin{aligned} k &= \frac{2(K_1 + W_{\text{other}} - mgy_2)}{y_2^2} \\ &= \frac{2[625{,}000 - 51{,}000 - (-58{,}000)]}{(-3.00)^2} = 1.41\times 10^5\text{ N/m} \end{aligned}$$

8.6

Conservative and Nonconservative Forces

When you throw a ball up in the air, it slows down as kinetic energy is converted into potential energy. But on the way down, the conversion is reversed, and the ball speeds up as potential energy is converted back to kinetic energy. If there is no air resistance, the ball is moving just as fast when you catch it as when you threw it.

If a glider moving on a frictionless horizontal air track runs into a spring bumper at the end of the track, the spring compresses and the glider stops. But then it bounces back, and if there is no friction, the glider has the same speed and kinetic energy it had before the collision. Again, there is a two-way conversion from kinetic to potential energy and back.

In both cases we find that we can define a potential energy function so that the total mechanical energy, kinetic plus potential, is constant or *conserved* during the motion.

A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a ***conservative force***. We have seen two examples of conservative forces: the gravitational force and the spring force.

The work done by a conservative force always has these properties:

1. It can always be expressed as the difference between the initial and final values of a *potential energy* function.
2. It is reversible
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

When the only forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

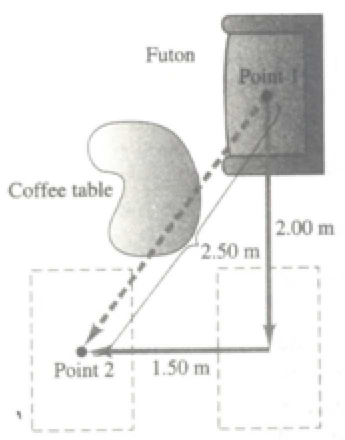
Consider the friction force acting on the crate sliding on a ramp. When the body slides up and then back down to the starting point, the total work done on it by the friction force is not zero. When the direction of motion reverses, so does the friction force, and friction does negative work in both directions. When a car with its brakes locked skids across the pavement with decreasing speed (and decreasing kinetic energy), the lost kinetic energy cannot be recovered by reversing the motion or in any other way, and mechanical energy is not conserved. There is no potential energy function for the friction force.

In the same way, the force of fluid resistance is not conservative. If you throw a ball up in the air, air resistance does negative work on the ball while it is rising and while it is descending. The ball returns to your hand with less speed and less kinetic energy than when it left, and there is no way to get back the lost mechanical energy.

A force that is not conservative is called a ***nonconservative force***. The work done by a nonconservative force cannot be represented by a potential energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a ***dissipative force***. There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding fire cracker fly of with very large kinetic energy, due to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. The spontaneous reassmbly of fragments into a complete firecracker is some thing that doesn't seem to be possible.

Example 1

You are rearranging your furniture and wish to move a 40.0-kg futon 2.50 m across the room.



However, the straight-line path is blocked by a heavy coffee table that you don't want to move. Instead, you slide the futon in a dogleg path over the floor; the doglegs are 2.00 m and 1.50 m long. Compared to the straight-line path, how much more work must you do to push the futon in the dogleg path? The coefficient of kinetic friction is 0.200.

Solution

Here work is done both by you and by the force of friction, so we must the energy relation that includes forces other than elastic or gravitational forces. We will use this relation to find a connection between the work you do and the work done by friction.

Figure above shows the initial and final points. The futon is at rest at both point 1 and point 2, so $K_1 = K_2 = 0$. The gravitational potential energy does not change because the futon moves only horizontally: $U_1 = U_2 = 0$. Therefore, from the equation $K_1 + U_1 + W_{other} = K_2 + U_2$ it follows that

$$W_{other} = 0$$
$$W_{you} + W_{friction} = 0.$$
$$W_{you} = -W_{friction}$$

Thus to determine W_{you}, we will calculate the work done by friction. Because the floor is horizontal, the normal force on the futon equals it weight *mg*, and the magnitude of the friction force is $f_k = \mu_k n = \mu_k mg$. The work you must do over each path is then:

Straight-line path

$$W_{you} = -W_{fric} = -(-f_k s) = +\mu_k mgs$$
$$= (0.200)((40.0)(9.80)(2.50) = 196\,J$$

Dogleg path

$$W_{you} = -W_{fric} = -(-f_k s) = +\mu_k mgs$$
$$= (0.200)((40.0)(9.80)(2.00 + 1.50) = 274\,J$$

The extra work you must do is 274 J – 196 J = 78 J

8.7

Potential Energy and Force

For a body with mass m in a uniform gravitational field, the gravitational force is $F_y = -mg$. We found that the corresponding potential energy is $U(y) = mgy$. To stretch a spring by a distance x, we exert a force equal to $+kx$. By Newton's third law the force that an ideal spring exerts on a body is opposite this, $F_x = -kx$. The corresponding potential energy function is $U(x) = \frac{1}{2}kx^2$.

We will encounter situations in which you are given an expression for the potential energy as a function of position and have to find the corresponding force. In electricity, it is often far easier to calculate the electric potential energy first, then determine the corresponding electric force afterward.

In the following, we find the force that corresponds to a given potential energy expression. First let us consider motion along a straight line, with coordinate x. We denote the x-component of force, a function of x, by $F_x(x)$, and the potential energy as $U(x)$. In any displacement, the work W done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U$$

Let us apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\Delta x$. This is approximate because $F_x(x)$ may vary little over the interval Δx. It is approximately true that,

$$F_x(x)\Delta x = -\Delta U$$
$$F_x(x) = -\frac{\Delta U}{\Delta x}$$

We take the limit as $\Delta x \to 0$; in this limit, the variation of $F_x(x)$ becomes negligible, and we have the exact relation

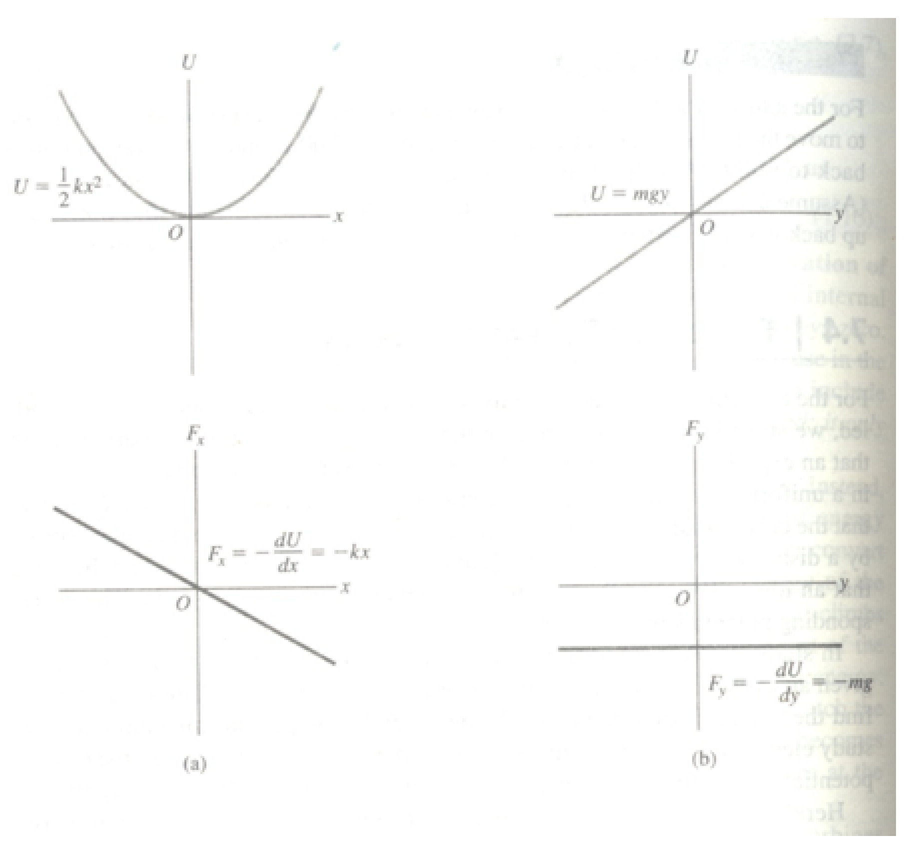
$$F_x(x) = -\frac{dU(x)}{dx}$$

In regions where $U(x)$ changes most rapidly with x (that is where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds large force magnitude. Also, when $F_x(x)$ is in the positive x-direction, $U(x)$ decreases with increasing x. So, $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of the above equation is that a conservative force always acts to push the system toward lower potential energy.

For an illustration, let us consider the function for elastic potential energy $U(x) = \frac{1}{2}kx^2$.

$$F_x(x) = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$$

which is the correct expression for the force exerted by an ideal spring (figure a).



Similarly, for gravitational potential energy we have $U(y) = mgy$. We get $F_y = -\frac{dU}{dy} = -\frac{d}{dy}(mgy) = -mg$, which is the correct expression for gravitational force (figure b).

Example

An electrically charged particle is held at rest at the point $x = 0$, while a second particle with equal charge is free to move along the positive x-axis The potential energy of the system is

$$U(x) = \frac{C}{x}$$

where C is a positive constant that depends on the magnitude of the charges. Derive an expression for the x-component of the force acting on the movable charge, as a function of its position.

Solution

From calculus, we know the derivative with respect to x of the function $1/x$ is $-1/x^2$. So the force on the movable charge for $x > 0$ is

$$F_x(x) = -\frac{dU(x)}{dx} = -C\left(-\frac{1}{x^2}\right) = \frac{C}{x^2}$$

The x-component of force is positive, corresponding to a repulsive interaction between like electric charges. The potential energy is very large for small x and approaches zero as x becomes large; the force pushes the movable charge toward large positive value of x, for which the potential energy is less. The force varies as $1/x^2$; it is small when the particles are far apart (large x) but becomes large when the particles are close together (small x).

Force and Potential energy in Three Dimensions

We can extend this analysis to three dimension, where the particle may move in the $x, y,$ or z-direction, or all at once, under the action of a conservative force that has components $F_x, F_y,$ and F_z. Each component of force may be a function of the coordinates $x, y,$ and z. The potential energy function U is also a function of all three space coordinates. We can now find each component of force. The potential energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x\Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relation

$$F_x = -\frac{\Delta U}{\Delta x}$$

The y, and z-components of force are determined in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}$$

To make these relations exact, we need to take the limits $\Delta x \to 0, \Delta y \to 0,$ and $\Delta z \to 0$ so that these ratios become derivatives. Because U may be a function of all three coordinates, we need to remember that when we calculate each of these derivatives, only one coordinate changes at a time. We compute the derivative of U with respect to x by assuming that y and z are constant and only x varies, and so on. Such a derivative is called a partial derivative. The usual notation is $\partial U/\partial x$ and so on; the symbol ∂ is a modified d to remind us of the nature of this operation. So we write

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

We can use unit vectors to write a single compact vector expression for the force $\vec{F}$:

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$$

The expression inside the parentheses represents a particular operation on the function U, in which we take the partial derivative of U with respect to each coordinate, multiply by the corresponding unit vector, and take the vector sum. This operation is called the gradient of U and is often abbreviated as ∇U. Thus the force is the negative of the gradient of the potential energy function:

$$\vec{F} = -\vec{\nabla} U$$

For an illustration, we consider the gravitational potential energy function $U = mgy$:

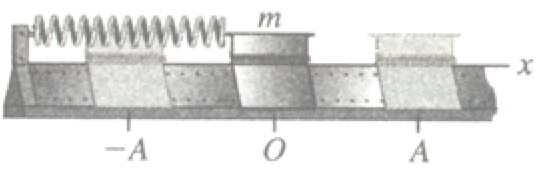
$$\vec{F} = -\vec{\nabla}(mgy)$$
$$= -\left(\frac{\partial(mgy)}{\partial x}\hat{i} + \frac{\partial(mgy)}{\partial y}\hat{j} + \frac{\partial(mgy)}{\partial z}\hat{k}\right) = (-mg)\hat{j}$$

This is just the familiar expression for the gravitational force.

8.8

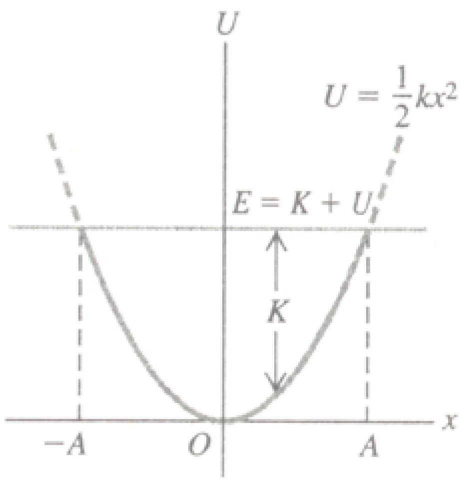
Energy Diagrams

Figure below shows a glider with mass *m* that moves along the *x*-axis on an air track.



The spring exerts on the glider a force with x-component $F_x = -kx$. Figure below is a graph of the corresponding potential energy function $U(x) = \frac{1}{2}kx^2$.

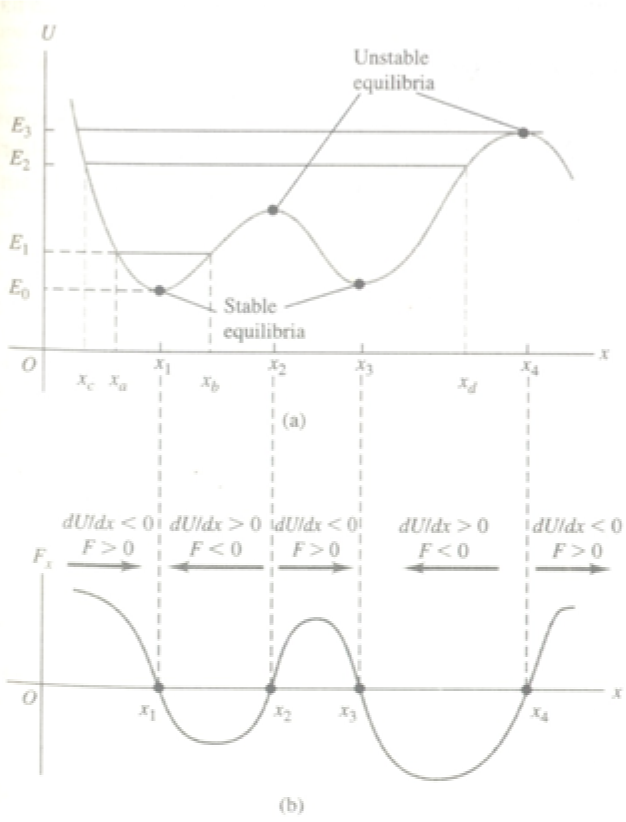
If the elastic force of the spring is the only horizontal force acting on the glider, the total mechanical energy $E = K + U$ is constant, independent of x. A graph of *E* as a function of x is thus a straight horizontal line.



The vertical distance between the *U* and *E* graphs at each point represents the difference *E* – *U*, equal to the kinetic energy *K* at that point. We see that *K* is greatest at x = 0. It is zero at the values of x where the two graphs cross, labeled A and –A in the diagram. Thus the speed *v* is the greatest at *x* = 0, and it is zero at $x = \pm A$, the points of *maximum* possible displacement from x = 0 for a given value of the total energy *E*. The potential energy can never be greater than the total energy *E*; if it were, *K* would be negative, and that's impossible. The motion is back-and-forth oscillation between the points x = A and x = −A.

At each point, the force F_x on the glider is equal to the negative of the slope of the $U(x)$ curve: $F_x = -dU/dx$. When the particle is at x = 0, the slope and the force are zero, so this is an *equilibrium position*. When x is positive, the slope of the $U(x)$ curve is positive and the force F_x is negative, directed toward the origin. When x is negative, the slope is negative and F_x is positive, again toward the origin. Such a force is sometimes called a *restoring force*; when the glider is displaced to either side of x = 0, the resulting force tends to "restore" it back to x = 0. An analogous situation is a marble rolling around in a round-bottomed salad bowl. We say that x = 0 is a point of ***stable equilibrium***. More generally, any minimum in a potential energy curve is a stable equilibrium position.

Figure (a) below shows a hypothetical but more general potential energy function $U(x)$. Figure (b) shows the corresponding force $F_x = -dU/dx$.



Points x_1 and x_3 are stable equilibrium points. At each of these points, F_x is zero because the slope of the $U(x)$ curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. The slope of the $U(x)$ curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the $U(x)$ curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called ***unstable equilibrium*** points; any maximum in a potential energy curve is an unstable equilibrium position.

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of E_1 and U graphs. Again U cannot be greater than E_1 because K cannot be negative. We speak of the particle as moving in a potential well, and x_a and x_b are the turning points of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c to x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_0 represents the least possible total energy the system can have.